Using chaos to keep period-multiplied systems in phase

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Periodically driven nonlinear systems can exhibit multiple-period behavior (period-2, period-3, etc.). Several such systems, when driven with the same drive, can be on identical attractors but remain out of phase with each other (e.g., one drive cycle for a period-doubled set of systems). This means that the basins of attraction for multiple-period systems can be divided into domains of attraction, one for each phase of the motion. A period-n attractor will have n domains of attraction in its basin. This out-of-phase situation is stable—small perturbations will not succeed in getting the systems in phase. We show that one can often use an almost periodic driving signal (generated from various chaotic systems) which will simultaneously (1) keep the motion of the systems nearly the same as the periodic driving case, (2) keep the basin of attraction nearly the same, and (3) eliminate the n domains of attraction. In other words, there will be only one domain for the basin. This means that any number of such driven systems will always be in phase. We display this effect in simulations and actual electrical circuits, discuss the mechanism for this effect (which is most likely a crisis), and speculate on some applications of the technique.

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I. INTRODUCTION

Many nonlinear systems when driven with periodic signals have regimes of their parameters where they behave periodically, but with a period that is some multiple of the period of the driving signal [1-3]. This means that the systems take n driving periods (where n is some integer) to return to their starting points. For example, when $n=2$, the systems will have a period of repetition that is twice that of the driving signal. This latter case is often referred to as a period-doubled system. Many other period multiplicities are possible.

The driven system could be in several different phases at a given point in the drive cycle. There are n possible phases for a period-n behavior. Which phase the system is at will depend on where it started when the drive signal was turned on. The salient point is that if several identical nonlinear systems are being driven by a periodic signal (see Fig. 1) so that they are period multiplied they can be permanently out of phase with each other. This out-of-phase situation is stable.

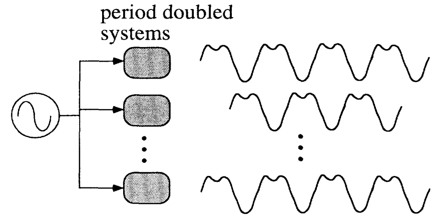

FIG. 1. Schematic of several multiple-period systems being driven by the same periodic drive. Depending on their initial conditions, each response system may be in or out of phase with any other.

For example, Fig. 2 shows the period-2 attractor and basin of attraction of a system (described below) which is similar to the Duffing system [3]. It is a system in which the x^3 Duffing term is approximated by a piecewise linear function. We will simply refer to it as the piecewise linear Duffing (PLD) system. A set of nearly identical such systems with the same drive will be in or out of phase by one drive cycle depending on which region (domain) of phase space they started in. Figure 3 shows out-of-phase signals from two PLD systems.

Furthermore, it is well known [4] that systems with

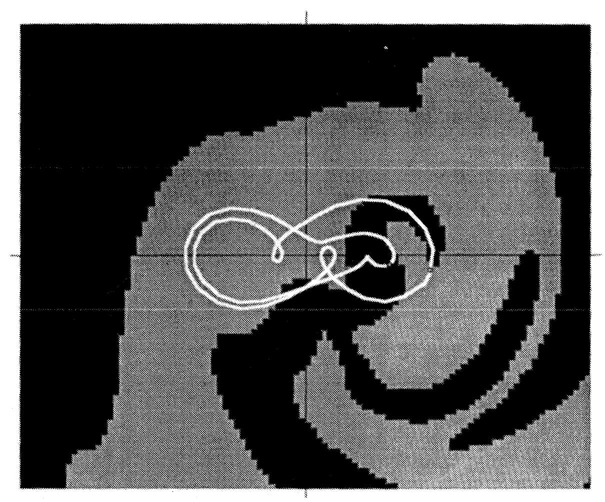

FIG. 2. Basin of attraction for the period-2 attractor PLD system (see text) along with the attractor. The black areas all synchronize with the point on the attractor in the black region. The gray areas all synchronize with the point on the attractor in the gray region.

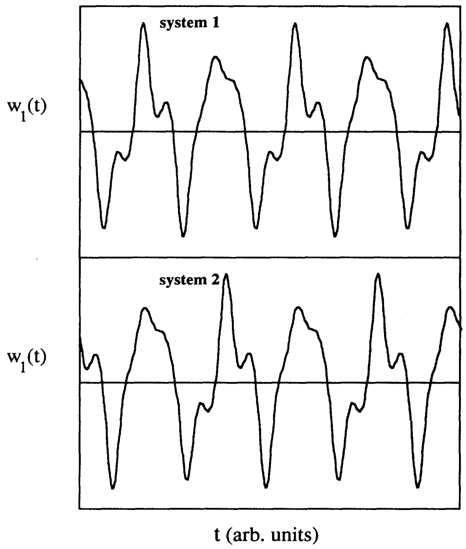

FIG. 3. Two out-of-phase signals $[w_1(t)]$ from two PLD systems started in different domains of Fig. 2.

several final attractors can have fractal basin boundaries. That is, the regions of phase space which mark where the systems started and in what attractor they will end up are complexly intertwined, in a fractal fashion. Here prediction of the final system state can be very difficult, as Yorke and co-workers [4] have pointed out, since the fractal structure gives an uncertainty to determining the domain of the initial conditions which is difficult to eliminate. In our present case of multiple-period attractors, this would mean that determining the final phase relations of the driven systems would become even more difficult if the domains have fractal boundaries.

There are instances when out-of-phase behavior is undesirable. However, there may be simultaneous instances when multiple-period behavior is required (perhaps because of the shape of the wave form needed) or unavoidable. We address the problem of maintaining a stable system motion close to the original multiple-period system's while keeping all such systems in phase (synchronized). Although this may seem like an artificial problem we note that there may be several areas where such criteria may appear. Examples are robotics, laser arrays, frequency dividers, or physiology. In robotics, if one is using nonlinear materials, devices that need to be driven by a timing signal, then any parts that have multiple-period behavior could get out of phase. In lasers, if one is operating a set of lasers in a period-doubled regime to get a half-frequency component, then these should be in phase to maximize output intensity [5]. In physiology, many tissues and organs are nonlinear and multiple behavior might be possible. In order to keep such systems in synchronization with each other in many driven circumstances we may find ourselves with the above in-phase requirements.

We have shown [6,7] that multiple-period systems can be driven to remain in nearly the same behavioral patterns, but be stable and in phase. To do this we alter the periodic drive to one that is partly or even totally chaotic. We call this new drive *pseudoperiodic,* since it stays close to the original periodic drive for periods of time, but it is never really periodic. Chaos is useful for creating such a drive because it has no periodicities. The success of such a new drive would mean that in Fig. 2 we would only have one domain of attraction (all black, for example) since all initial conditions would lead to synchronization (all in phase). In other words, what we have really done is eliminate the multiple domains that accompany period-n behavior. For systems which have fractal basins of attraction, boundaries between multiple-period domains will vanish and to some extent the prediction problem of McDonald *et al.* [4] will be diminished or eliminated.

Below in Sec. II we show approaches to building such pseudoperiodic drives. There are several and which is best will depend on the particular system. In some cases using noise in place of chaos will also work. We comment on the efficiency of each type of drive. In Sec. III we demonstrate these principles in a circuit. The circuit results lead to an explanation of why pseudoperiodic driving works. This is explored further in Sec. IV using numerical simulations where we make connections to other bifurcations and crises. We conclude with several remarks and speculations about the application of pseudoperiodic driving.

II. BUILDING A PSEUDOPERIODIC DRIVE

In this and the following sections we refer to the driving system as simply the *drive* and the driven system as the *response.*

The basic idea is to build an alternative drive that is not too different from the original periodic drive. There are several approaches to this. Which is best to use will depend on the response system. The three criteria to be satisfied are (i) the response system must remain stable, (ii) the pseudoperiodic drive must be similar to the original periodic drive, and (iii) the present drive must eliminate out-of-phase behavior (i.e., all response systems are in synchronization). In the following sections we will show examples of several types of pseudoperiodic drives and their effects.

Some possible approaches to constructing a pseudoperiodic drive follow.

(1) Direct replacement of the periodic drive with a chaotic one which has pronounced spectral peaks at the same frequency as the periodic drive. There are chaotic systems which are oscillatory in nature, such as the Rössler system [8]. This system is described by the equations

$$\begin{aligned} \dot{x} &= -\gamma(y+z) , \\ \dot{y} &= \gamma(x+ay) , \\ \dot{z} &= \gamma[b+z(x-c)] . \end{aligned} \tag{1}$$

In certain parameter regimes it is chaotic, but the chaos has a strong periodic quality. The γ parameter can be tuned to have a spectral peak at the same frequency as the original driving system. Figure 4 shows a Rössler

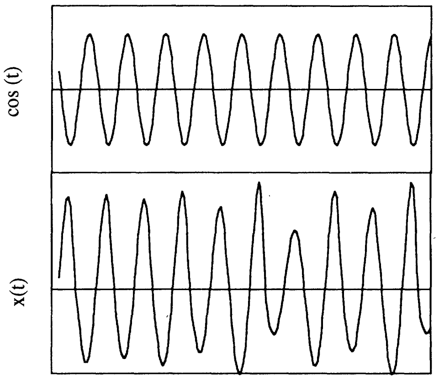

FIG. 4. Rössler vs sine wave.

$x(t)$ time series compared to a cosine of the same frequency. Figure 5 shows the Fourier spectra of both signals demonstrating the similarity of the Rössler to the sinusoid. Furthermore, periodically driven systems which are chaotic often have spectral features similar to their drives and are also candidates for use as replacement pseudoperiodic drives. For example, certain cases of cosine-driven, mildly chaotic Duffing systems show time series with spectral similarities to their drive. As we show below, it appears that direct replacement of a periodic drive by its chaotic counterpart, if possible, often leads to the most robust pseudoperiodic drive when compared to the following methods.

(2) Partial replacement of the periodic drive by adding in some small amount of chaotic signal. This would be described by a linear combination drive, $rS_{\text{periodic}}+sS_{\text{chaotic}}$. We have often used $r=1.0$ and $s=\epsilon$, but one can also take a homotopic approach and vary the signal from pure periodic ($r=1.0, s=0.0$), to some middle case ($r=\alpha, s=1-\alpha, |\alpha|<1.0$), to pure chaotic ($r=0.0, s=1.0$), depending on the stability and quality of the response time series. Sources of chaotic signals can be systems from (1) above, or more broadband chaotic systems.

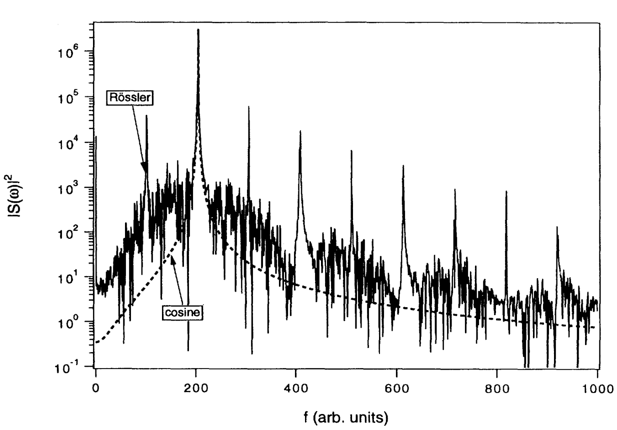

FIG. 5. Spectra of Rössler vs sine wave from 4096-point fast Fourier transforms of each signal.

(3) Partial replacement of the periodic drive by adding in some small amount of noise signal. This would be described by a linear combination drive, $rS_{\text{periodic}}+sS_{\text{noise}}$, as in (2). In general, we find noise to not be as efficient as chaos, but this will surely depend on one's response.

(4) Modulation of a periodic drive by a chaotic or noise sources. This would be described by a multiplication drive, $(r+S_{\text{chaotic}})S_{\text{periodic}}$, where r is a constant (possibly zero). Whether the chaotic signal used is zero mean or not will depend on the response system.

(5) Chaotic parametric modulation of the response system [for example, the parameter k in Eq. (7)]. In cases where the parameter modulated is not added to the drive or multiplying the drive, this is different from cases (2) and (4). One could vary the parameter slightly by adding some chaos to the parameter, viz., $k+\epsilon x(t)$, where $x(t)$ is from a chaotic system. This will sometimes work with noise modulation, too.

(6) Quasiperiodic driving. This is a case considered by Heagy and Ditto [9] and Heagy and Hammel [10] in a slightly different context. As we show below the mechanism for the bifurcations seen in these papers is related to the behavior described here. A general problem with using quasiperiodic driving is that, experimentally, it is difficult to keep the drive quasiperiodic. Slight variations in frequency of the drive may transform it from quasiperiodic to periodic (albeit one with, perhaps, a high period). This leads to the possibility of having multiple-period behavior again, which was to be avoided.

We now demonstrate how to implement some of these ideas in nonlinear circuits.

III. CIRCUIT DEMONSTRATIONS

To test multiple domain elimination using pseudoperiodic driving on a real system, we built several closely matched nonautonomous nonlinear circuits. We discuss the relevant typical examples here.

A. The response circuit

For the response circuits we considered a circuit similar to a Duffing system with a piecewise linear circuit element emulating the cubic term, the PLD system. The equations describing these circuits are

$$\frac{dy}{dt}=\alpha[A\cos(\omega t)+C-0.2y-G(x)]\ , \tag{2}$$

$$\frac{dx}{dt}=\alpha y\ , \tag{3}$$

$$G(x)=\begin{cases}0 \quad [\text{abs}(x)<1.2] \\ x-1.2\,\text{sgn}(x) \quad [1.2\le \text{abs}(x)<2.6] \\ 2x-3.8\,\text{sgn}(x) \quad [2.6\le \text{abs}(x)]\ ,\end{cases} \tag{4}$$

where A is the amplitude of the cosine drive and C is a constant offset that may be added to the drive. The time factor α is $1\times10^4\ \text{s}^{-1}$. We use the function $G(x)$ instead of x^3 because it is easier to closely match two response circuits using a piecewise linear function. The function

x^3 requires the use of analog electronic multiplier chips, whose characteristics vary from chip to chip. Figure 6 is a schematic of the PLD circuit. Figure 7 is a schematic of the piecewise linear circuit which executes the function $G(x)$. The piecewise linear circuit used many 20 turn trimpots so that two of these circuits could be closely matched. We also used trimpots in series with resistors R6 and R9 in Fig. 6 to match the two PLD circuits.

B. Sources for pseudoperiodic drive signals in the experiments

As a chaos source we used an unstable oscillator with a hysteritic element [11,12]. The equations modeling this circuit are

$$\frac{dx_1}{dt}=10^3(1.5x_2+2.2x_1+2.2x_3)\ ,$$

$$\frac{dx_2}{dt}=10^3x_1\ ,$$

$$\epsilon\frac{dx_3}{dt}=(1-x_3^2)(Sx_1-D+x_3)-\delta_3x_3\ ,$$

where $\epsilon=0.3$, $S=1.667$, $D=0.0$, and $\delta_3=0.001$. The x_3 equation is a phenomenological equation used to model the hysteresis. An extra damping term may be needed in

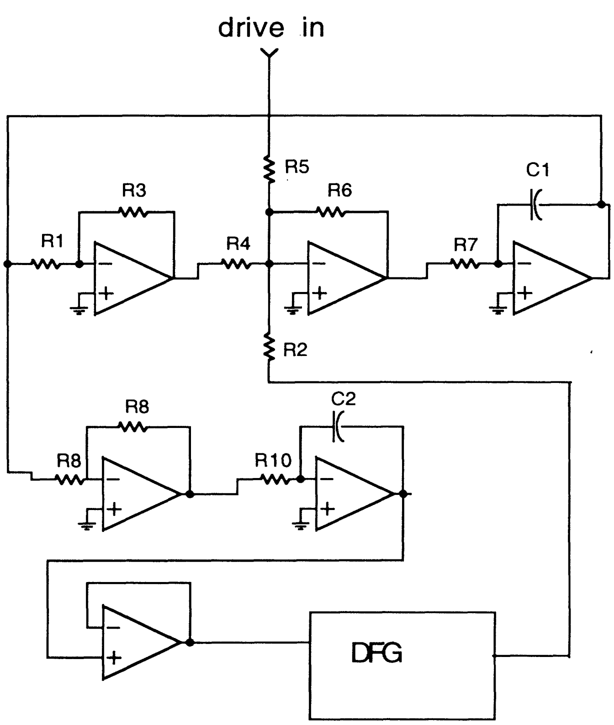

FIG. 6. Duffing equation circuit used to test period-doubling synchronization through pseudoperiodic driving. R1=R3=R4=R5=R6=10 kΩ, R2=39.2 kΩ, R7=R10=100 kΩ, R8=R9=1 MΩ, C1=C2=0.001 μF. All operational amplifiers are type 741. The square marked DFG is a diode function generator that provides a piecewise linear approximation to the x^3 term in the Duffing equations.

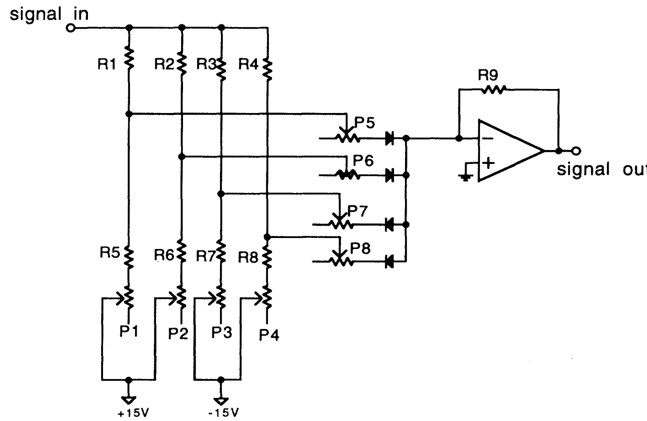

FIG. 7. Diode function generator circuit to approximate x^3 term in Duffing circuit. R1=R2=R3=R4=R9=100 kΩ. R5=R7=680 kΩ. R6=R8=2 MΩ. P1=P3=20 kΩ potentiometer. P2=P4=50 kΩ potentiometer. P5=P6=P5=P8=20 kΩ potentiometer in parallel with 100-Ω resistor. The diodes are all type 1N485B. The potentiometers are used to match different x^3 circuits to each other. The amplifier is type 741.

the x_2 equation to simulate the effect of losses in the circuit.

We also used a General Radio 1390-B noise generator to provide white noise with a spectrum that was flat to 20 kHz, and a Hewlett Packard function generator to generate a sine wave with a frequency of about 400 Hz.

C. Results for various combinations of drive and response circuits

We studied the synchronization of two period-doubled PLD circuits described above. We first ran these circuits with a drive amplitude of $A=5.19$ V and an offset of $C=0.3$ V. We added chaos from the hysteretic oscillator circuit or noise from the white noise generator to the drive signal.

For both types of drive we found a fairly sharp threshold for synchronization when the multiplier ϵ of the signal added to the drive was 0.08. Above this threshold, the two circuits were synchronized 90% of the time. The threshold was sharper than for the sub-Rössler circuits used in our previous work [6] because the PLD circuits were easier to match. This threshold varied as the drive amplitude varied, becoming smaller near points where bifurcations in the PLD circuit response occurred. We saw similar results when we used a periodic signal with a frequency incommensurate with the drive signal.

Figure 8(a) shows a period-doubled attractor for the PLD circuit driven with a sine wave, while Fig. 8(b) shows the same circuit with just enough chaos added to the sinusoidal drive to synchronize two systems.

We observed that one or the other (or both) of the circuits pass through an unstable period-1 orbit just before they synchronize. When the two circuits are initially out of phase, they may travel around the period-1 orbit a different number of times. If one circuit makes an even

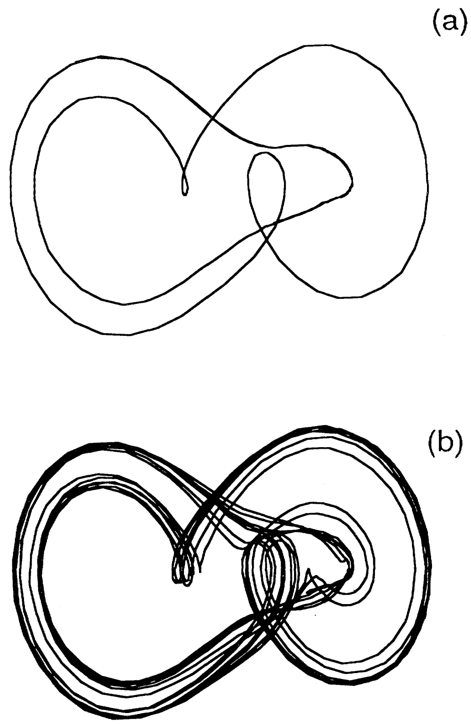

FIG. 8. Period-doubled attractors (x vs y) for Duffing circuit with periodic or pseudoperiodic driving. (a) is for a Duffing circuit driven periodically, while in (b) just enough chaos has been added to the periodic drive to keep two period-doubled circuits synchronized.

number of period-1 orbits before returning to the period-2 and one makes an odd number, the two circuits will be synchronized. If the two circuits are already synchronized, they will both make the same number of period-1 orbits, and will stay in synchronization. Figure 9 shows the x signal from each of two circuits. The two signals are initially out of phase. The signal on top then goes through three period-1 orbits, so that when it returns to the period 2, it is in phase with the signal on the bottom.

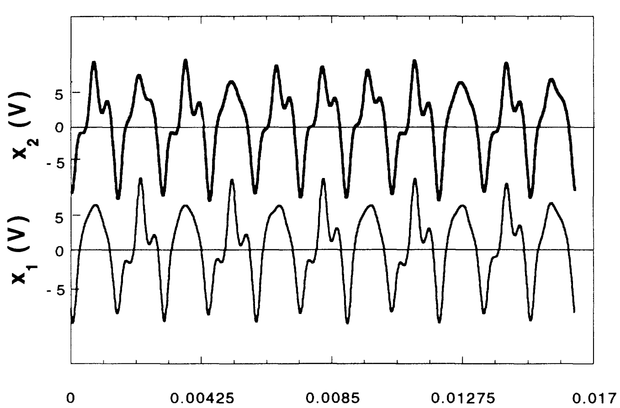

FIG. 9. Time series for signals corresponding to the x variables in two nearly identical piecewise linear Duffing circuits driven with a pseudoperiodic signal. Both time series are initially period doubled but out of phase by one drive cycle. The x_2 signal then goes through three unstable period-1 cycles starting at about $t = 0.006$ s, causing its phase to flip.

We then varied the amplitude of the drive signal for the period-doubled PLD system and observed how this changed the threshold for synchronization. Figure 10 shows the results of this investigation when chaos from the hysteretic oscillator circuit or white noise was added to the sinusoidal drive signal. In this figure, the value of the multiplier ϵ has been normalized so that the largest value is 1.

The sinusoidally driven PLD system undergoes a bifurcation from period 1 to period 2 at a drive amplitude of 4.15 V. In Fig. 10, the multiplier ϵ is smallest for drive amplitudes just above the period-doubling value, and increases as the drive amplitude is increased. The threshold value of ϵ reaches a peak whose location depends on the type of driving used. The threshold ϵ decreases as the drive amplitude increases above this point, becoming very small at 7.2 V drive. At 7.25 V, the sinusoidally driven PLD circuit becomes chaotic.

To demonstrate the effect of having nonidentical systems, we varied one resistor in one of the period-doubled PLD circuits and measured the fraction of time that the two systems were synchronized as a function of ϵ when the drive amplitude was 5.19 V. Figure 11 plots this relation when the resistor R4 was replaced with a 9-kΩ resistor (10% parameter difference) and a 7.5-kΩ resistor (25% parameter difference). When the two circuits were matched as closely as possible, the parameter difference is still approximately 1%.

In Fig. 11 the circuits with 1% parameter difference and 10% difference synchronize in approximately the same fashion, although just above the threshold for synchronization the 1% difference circuits are in synch-

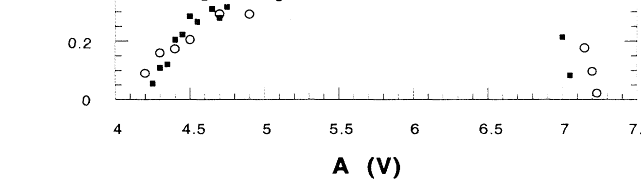

FIG. 10. Comparison of the threshold for synchronization of pseudoperiodically driven period-doubled piecewise linear Duffing circuits. The amplitude of the periodic part of the drive is A. For the open circles, the pseudoperiodic drive was made by adding chaos to the periodic drive, while for the dark squares, white noise was added. To correct for different initial amplitudes of the noise or chaos, ϵ_{n1} is the value of ϵ normalized so that the largest value was 1.

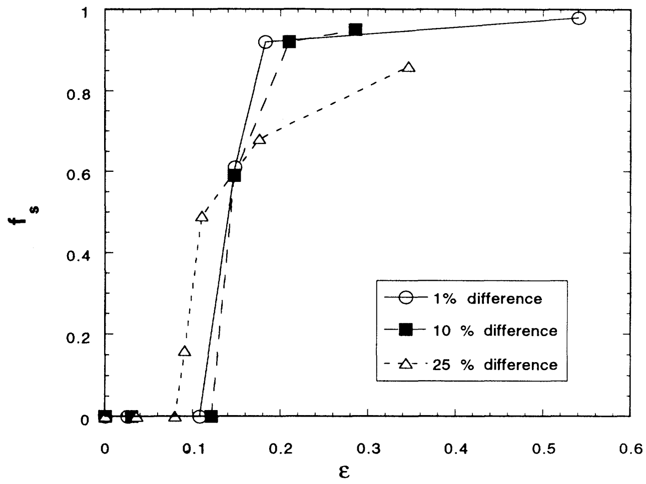

FIG. 11. Comparison of synchronization threshold for pseudoperiodically driven piecewise linear Duffing circuits when resistor R4 in one circuit was changed to create a parameter difference. The fraction of time the circuits are synchronized is f_s. The open circles represent a parameter difference of 1%, the dark squares represent a difference of 10%, and the open triangles represent a difference of 25%.

ronization a greater fraction of the time than the 10% circuits. The synchronization threshold appears to be lower for the 25% parameter difference circuits. This is probably because such a large parameter change coincidently reduces the synchronization threshold in the circuit with the modified resistor. The fraction of the time that the 25% circuits are synchronized does not rise as quickly above the threshold as it does for the 1% and 10% circuits. When the 1% and 10% circuits are in synchronization 90% of the time, the 25% circuit is in synchronization only 70% of the time.

For well-matched circuits, if the two circuits are in phase, then when one goes through a period-1 orbit and changes its phase, the other will most likely do the same. When the circuits are not well matched, there is a much larger chance that when started in phase, one will change phase while the other does not. This is because parameter differences result in degradation of synchronization of any two similar, stable systems [11].

IV. ANALYSIS OF A PSEUDOPERIODICALLY DRIVEN SYSTEM

A. Prelude

In Ref. [6] we give a heuristic argument as to why one might expect that adding a chaotic signal to a periodic drive might eventually lead to the destruction of multiple periodic behavior. This argument shows that if we change the drive of a system slightly, we expect the system at first to have behavior close to its original behavior. But as we increase the chaos in the drive (i.e., change the drive more) we should reach a point at which the response is no longer near its original trajectory and the behavior is qualitatively different. Thus we would predict a threshold above which multiple-period trajectory is no longer stable. This threshold should be seen when we use techniques like (2) in Sec. II and was seen (above) in circuit implementations of pseudoperiodic technique (2). Whether the new motion is stable and still similar to the original is another matter which we take up below.

Mathematically we can understand the threshold argument as follows. If the original system is given by

$$\dot{w} = f(w,v) , \tag{5}$$

where v is the driving signal and the behavior is multiple period (nonchaotic), then changing v to another drive signal v' will lead to the variational equation

$$\Delta\dot{w}' = D_w f(w',v) + B(t) , \tag{6}$$

where $\Delta w = w' - w$, $B(t) = f(w,v') - f(w,v)$, and $D_w f$ is the Jacobian of the vector field. If v' is in some sense close to v, then $B(t)$ will be small (in some norm) and, using the transfer function [6], we can show that Δw remains small. This follows from the smallness of $B(t)$ and the stability of the original system (as realized through $D_w f$). However, as v' is changed to be greatly different from v, $B(t)$ will no longer remain small and we expect some level of v' at which the new motion no longer remains near the original motion step by step in time.

The surprise is that if v' is still not too different from v, the new motion will be on an attractor very similar to the original and it can be stable. However, the multiple periodicity is lost and only one domain remains. This loss of domains comes about apparently by a type of bifurcation called a crisis. We explain this below after introducing a model to display the behavior and help us analyze the threshold and multiple domain destruction.

B. Numerical simulation

We chose a model of the above PLD system as the response. The equations of motion are similar to Eqs. (8)–(10):

$$\begin{aligned} \frac{dw_1}{dt} &= w_2 , \\ \frac{dw_2}{dt} &= -kw_2 + w_1^3 + \alpha v + \beta , \end{aligned} \tag{7}$$

where v is the drive, either sinusoidal [$\cos(\omega t)$] or pseudoperiodic. The cubic term w_1^3 was approximated by the piecewise linear function $G(x)$ in Eq. (4). We set $k=0.2$, $\alpha=4.19$, $\beta=0.3$, and $\omega=0.422\,23$, all of which match the parameters of the circuit. A small amount of smoothing using local convolution was done in a region about the kinks in $G(x)$ corresponding to a size of $\Delta x \approx 0.08$. The motion in this regime is period 2. The period-2 attractor bifurcates from the period 1 by a flip saddle bifurcation [13]. Figure 2 shows the attractor for this system and the domains of attraction for each phase of the period-2 motion. Figure 12 shows a detailed view of the period-2 attractor along with the unstable period-1 and simultaneous system points. This will be used for com-

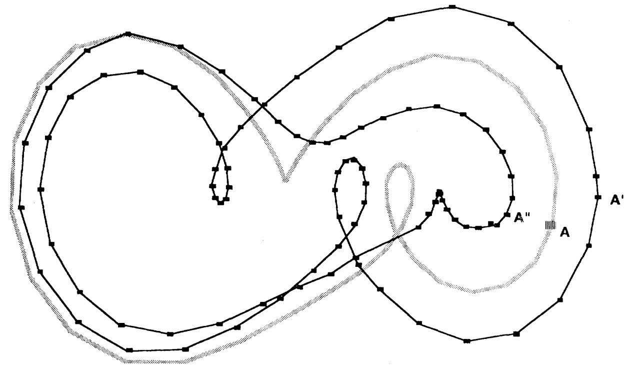

FIG. 12. Positions of out-of-phase period-2 points (A' and A'') relative to unstable period-1 (flip saddle points A).

parison with the pseudoperiodic driven cases below and in their analysis. The system actually has many of the same attractors as the original PLD system (see the attractors in Ref. [13]) in several parameter regimes.

For our chaos drives we tested two sources. One was the Rössler system Eq. (1) with $a=b=0.2$, $c=4.5$, and $\gamma=0.12$, the latter being set to provide a strong spectral peak at the same frequency as $\cos(\omega t)$. This was used to completely replace the $\cos(\omega t)$ drive as described in part (1) of Sec. II. Another chaos source was the Lorenz system [3]

$$\begin{aligned} \dot{x} &= \gamma\sigma(y-x) , \\ \dot{y} &= \gamma(-xz+rx-y) , \\ \dot{z} &= \gamma(xy-bz) , \end{aligned} \tag{8}$$

where $\gamma=0.04$, $\sigma=10.0$, $b=\frac{8}{3}$, and $r=60.0$. The Lorenz chaos was broadband and was used with the technique described in part (2) of Sec. II, addition of chaos to the $\cos(\omega t)$ drive.

In order to gauge time for transient behavior in terms of the typical system times we will refer to times in number of cosine periods. We will simply term these cycles as follows.

1. Rössler pseudoperiodic driving. Replacement of the drive

We replaced the $\cos(\omega t)$ with $x(t)$ from the Rössler drive. We scaled $x(t)$ so that it had a rms value equal to the cosine value of $1/\sqrt{2}$. We first chose the starting point for the Rössler so that $x(t)$ had its first zero crossing at the same time as the cosine. This was to mimic the cosine as much as possible, at least at first before Rössler and cosine would get out of phase (see Fig. 4). This turned out to be unnecessary as any starting phase of the Rössler worked as well as any other. The Rössler-driven PLD system turned out to be stable (negative conditional Lyapunov exponents [14]), so that one of our three criteria was fulfilled: stability.

We ran two PLD systems along with the Rössler. They were started one cosine drive cycle out of phase on the period-2 attractor. Typically, the Rössler drive caused one of the PLD systems to jump to a trajectory near the other in one or two cycles. Both PLD systems would then converge to within numerical integration accuracy (typically 10^{-4} times the attractor size) within five to seven cycles. Because of the stability they would stay converged. Along with calculating the Lyapunov exponents we tested stability by occasionally perturbing one of the responses and letting the integration run for long times (thousands of cycles). All tests showed stability of the convergence of both responses, so that another of our criteria is satisfied: in-phase or synchronized behavior. This means the basin of attraction now has only one domain so Fig. 2 would be all one "color."

Figure 13 shows the trajectories of the responses at the start of a typical run using Rössler driving (both PLD's started one cosine drive cycle apart). Note that after roughly one cycle both responses are on the same side of the unstable period-1 orbit and beginning to converge. We analyze this more in Sec. IV C.

Figure 14 shows the $w_1(t)$ signal from the PLD response. Note the similarity to the periodically driven period-2 system 2 case in Fig. 3. Some differences show up in exact heights of peaks and a few small details, but the basic shapes and behavior are retained. Figure 15 shows the attractor for the pseudoperiodically driven case and should be compared to Fig. 2. Although the pseudoperiodic case is "noisier" it shows that the behavior is close to the general behavior of the period-2 case, so that the remaining criterion for pseudoperiodic driving is satisfied: similarity to the original periodically driven case.

2. Lorenz pseudoperiodic driving. Addition to drive

We added chaos from the Lorenz system to create a pseudoperiodic drive according to $v(t)=\cos(\omega t)+\epsilon x(t)$, where $x(t)$ is the x component of the Lorenz system. By varying ϵ we can display the threshold phenomena and the effects of varying the amount of chaos in the drive.

We found that for this configuration of systems we needed $\epsilon>\epsilon_c=0.0076$. If we rescale the rms value of $x(t)$ so that it is $1\sqrt{2}$ we find that the threshold occurs at

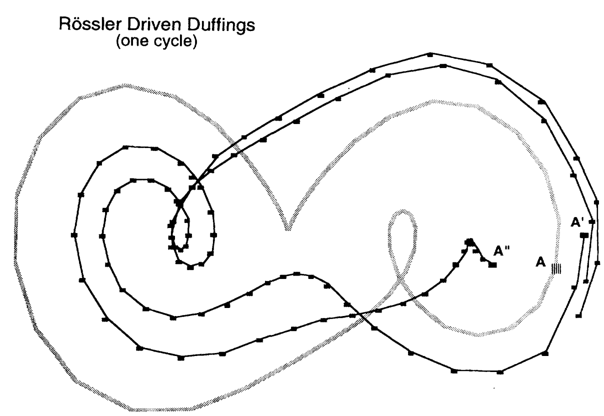

FIG. 13. Convergence of originally out-of-phase points (see Fig. 12) after crossing unstable period 1 when using a Rössler pseudoperiodic drive.

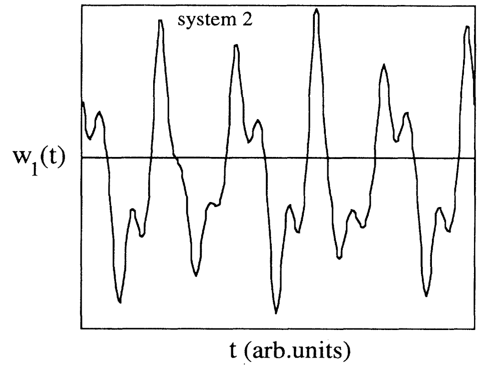

FIG. 14. Time series $w_1(t)$ for pseudoperiod-2 PLD (compare to Fig. 3).

$\sim 6\%$ (rms) chaos. Figure 16 shows the number of cycles to convergence of both PLD responses, where convergence is measured by integration accuracy as in the Rössler case. Below threshold no convergence took place in tests lasting for up to 20 000 cycles. Figure 17(a) shows the attractor of one of the PLD systems when $\epsilon=0.007$ ($\sim 5.5\%$ chaos, just below the threshold). Just above threshold the average number of cycles to synchronization begins at very high values and decreases rapidly, initially for ϵ increasing away from ϵ_c. The average number of cycles to synchronization then levels off.

The behavior above threshold is shown in Figs. 17(b) and 17(c). In Fig. 17(b) the ϵ value chosen will guarantee synchronization in about 20 cycles on average. At this ϵ value it takes on average 11 cycles for one of the responses to cross over the unstable period 1 and begin to converge to the other. This convergence is shown in Fig. 18 and is similar to the Rössler case except that it takes on average a few more cycles. This gives a somewhat "cleaner" attractor compared with the Rössler case, but in real applications where parameter differences will cause occasional out-of-synchronization behavior more rapid convergence to synchronization than this may be necessary. In order to match the rapidity of synchronization of the Rössler-driven case a value of $\epsilon > 0.05$ must be used. This does cause rapid convergence, but as can be seen in Fig. 17(c) the topology of the attractor is greatly degraded.

Which type of pseudoperiodic drive one uses is dependent on the requirements and on the response. Some systems will not be stable to complete replacement of the

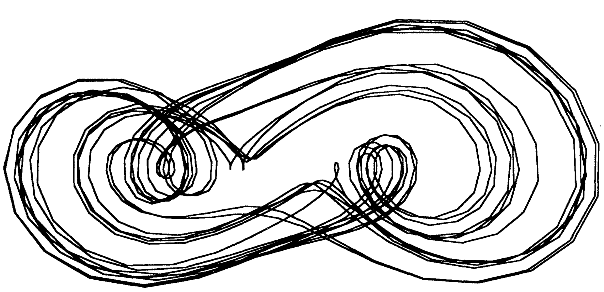

FIG. 15. PLD pseudoperiod-2 attractor.

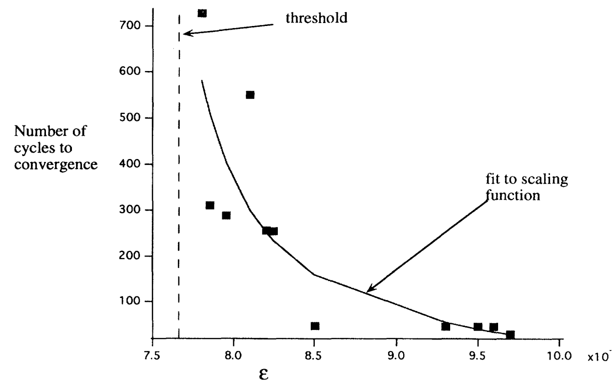

FIG. 16. Number of cycles to convergence of out-of-phase points (see Fig. 12) as a function of added Lorenz chaos in the pseudoperiodic drive for the PLD response. The threshold corresponding to a crisis is evident. The scaling function fit to the numerical data is $1/(\epsilon-\epsilon_c)^{\nu}$.

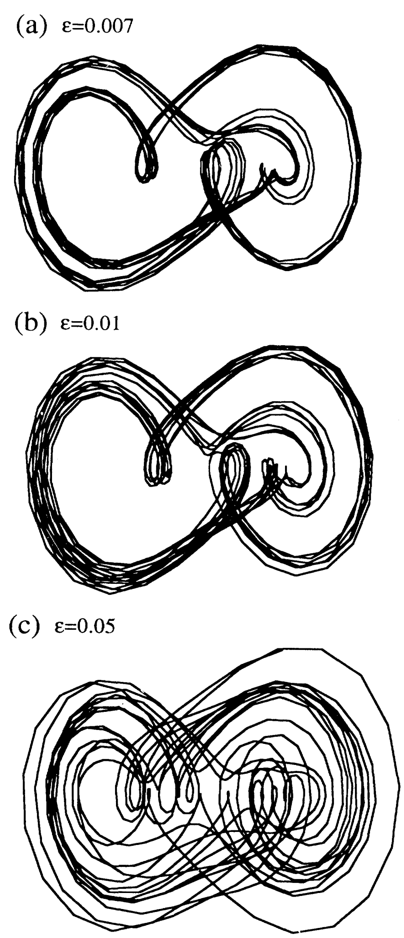

FIG. 17. PLD pseudoperiod-2 attractors (Lorenz-added-to-cosine pseudoperiodic drive), (a) $\epsilon=0.007$, (b) $\epsilon=0.01$, (c) $\epsilon=0.05$.

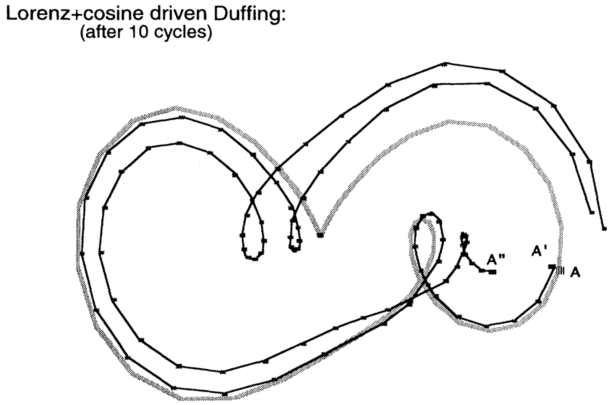

FIG. 18. Convergence of originally out-of-phase points (see Fig. 12) after crossing unstable period 1 when using a Lorenz-added-to-cosine pseudoperiodic drive.

periodic drive by a chaotic one like the Rössler [6]. What is interesting is that when the complete replacement technique works, it appears to be more robust than the additive method in the numerical modeling, but not necessarily in the experiment. The reasons for these differences are not yet understood.

C. Analysis of the threshold: A crisis

The threshold phenomenon is related to a crisis. In order to see this we examine the Poincaré map by strobing the system at times $t=2\pi n/\omega$ ($n=0,1,2,\ldots$), which causes the generation of a map of the points labeled A in Fig. 12. These starting points are shown in a closeup of this region of the Poincaré section in Fig. 19(a). The A point on the unstable period 1 is a saddle-node fixed point. The tangents to the stable and unstable manifolds are also shown in Fig. 19. The latter we calculated by finding the eigendirections of the Lyapunov exponents [14] for the saddle-node point.

If the driving were periodic, the points on the period-2 orbit (A' and A'' in Fig. 19) would merely alternate to opposite sides of the unstable direction. However, because of the added chaos the pseudoperiod-2 points map to points perturbed from their period-2 locations. When enough chaos is added so that the pseudoperiodic drive is above the threshold for synchronization, one of the points is finally perturbed over the stable manifold associated with the unstable period-1 point. This is shown in Fig. 19(b). From this point on, because of the relative location of the pseudoperiod-2 points to the stable and unstable manifolds the two responses will converge to the same attractor.

There is some chance that one of the pseudoperiod-2 points would be knocked back over to the other side of the unstable manifold, but this is small. Furthermore, as the two pseudoperiod-2 trajectories converge, if one is knocked over, the other is more and more likely to follow since they are both driven with the same pseudoperiodic signal. In the case of perfect convergence (both responses reach the exact same trajectory) the two responses never

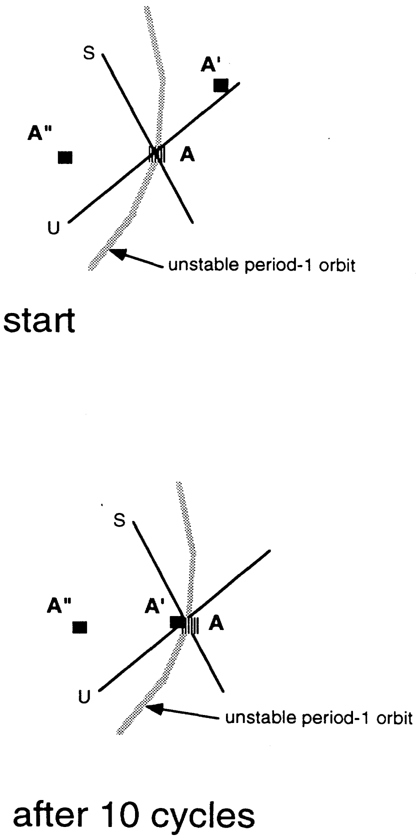

FIG. 19. Poincaré maps of out-of-phase points before (cycle 1) and just after the crisis point (cycle 11), showing the crossing of the stable manifold of the unstable period-1 point.

again get out of synchronization or phase. This is seen in the numerical studies. In real systems the two responses will never have the same parameter values and so will never converge to the same trajectory [14]. In this latter case one of the responses will occasionally get knocked out of phase with the other. Some time later they will be knocked in phase again. If the parameter differences are not too large the systems will still be in phase a large percentage of the time, e.g., $\sim 90\%$ for the circuits we studied.

This threshold behavior is similar in scenario to a phenomenon called a crisis [10,15]. In a crisis a dynamical system is operating below the threshold for a bifurcation (we are using the word bifurcation in the most general sense here as a topological change to the attractor). As the salient parameter for the bifurcation is moved closer to the threshold it is possible for the system to display behavior similar to that found above the threshold, but only for finite times. In some cases the above-threshold behavior recurs at random intervals (e.g., intermittency [16]). In others the above-threshold behavior only occurs during a transient (e.g., chaotic transients [17]) after which the below-threshold behavior persists.

The average time τ for the transients asymptotes to infinity as the system parameter (ϵ here) approaches the threshold [9,10,15,17–19]. Grebogi *et al.* have shown that these times should fit a scaling law of the form

$$\tau \sim \frac{K}{(\epsilon-\epsilon_c)^{\nu}} . \tag{9}$$

In Fig. 18 we show a fit to the convergence time for a law of this form. We find $\epsilon_c = 0.0074$ and $\nu = 0.9596$. In many of the simulations we find the scaling exponent to be close to 1.0. The fit suggests that we can view the pseudoperiodic threshold as a crisis with the amount of chaos in the drive ϵ being the bifurcation parameter. This is the view taken in [8,10]. There Heagy and coworkers add another periodic drive to a Duffing system [9] and a logistic map [10]. They suggest that this is a new type of bifurcation.

Another view of this phenomenon is that of a noise-induced crisis after the work by Sommerer *et al.* [18] and the theoretical work by Arecchi, Badii, and Politi [20]. In this view a system parameter is maintained at a value just before a crisis and the system is driven with noise. The noise will knock the system into a postcrisis state allowing it to behave similarly to the postcrisis orbit for some time τ before it falls back to its precrisis state. The theory states that the scaling behavior for τ for this type of phenomenon is

$$\tau \sim \epsilon^{-\gamma} g(|\alpha-\alpha_c|/\epsilon) , \tag{10}$$

where ϵ is the noise level, g is a function specific to the particular system, and α is the bifurcation parameter. In our case the noise comes from adding the chaos to make the pseudoperiodic drive and the bifurcation parameter can be any parameter which takes the system from period-1 to period-2 behavior. We chose α to be the amplitude of the $\cos(\omega t)$ drive.

In order to test the second view we numerically studied the pseudoperiodically driven PLD for $\alpha = 4.19$ and 2.5. The period-2 bifurcation comes at $\alpha_c \approx 2.04$. Figure 20 shows a plot of the average time to synchronization (transient time) for two period-doubled PLD systems. The axes have been scaled as in [18] to display the adherence of the system behavior to Eq. (10). As the figure shows, there appears to be scaling according to a noise-induced crisis model.

At this point we would conclude that the nature of the pseudoperiodic crisis has elements both like that observed in the quasiperiodic case by Heagy *et al.* and also

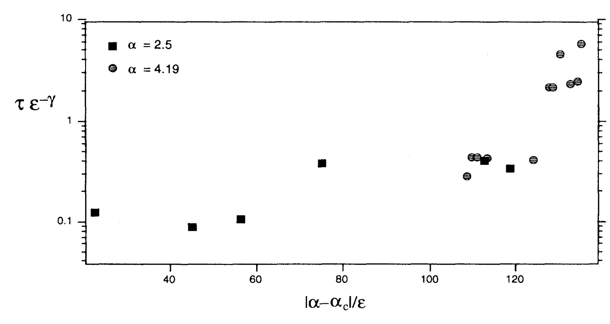

FIG. 20. Average scaled time to synchronization for noise driven scaling for two period-doubled PLD systems (cf. [18]).

like that observed by Sommerer *et al.* depending on which parameter one chooses as the bifurcation parameter (noise level ϵ or system parameter α). This would imply an underlying connection between these crises, perhaps treatable as a codimension-2 bifurcation. Our study of chaotically varying the logistic map [21] parameter shows that bifurcations very similar to those Heagy saw also come about in pseudoperiodically driven maps.

However, we note that the above analysis may not apply to all pseudoperiod drives. In the case of chaos added to a periodic drive [method (2) of Sec. II] we retain an underlying drive component whose phase is constant with respect to the original drive $[\cos(\omega t)]$. But in the use of complete replacement of the drive by a chaotic one with strong spectral components similar to the original drive [method (1) of Sec. II] there is no constant phase or true underlying periodic signal, although if there is only weak chaos, then the drive "phase" does not wander much from the periodic phase and there may actually be some average phase that can be used for reference. But generally, we cannot just treat this as a case of "noise" on top of a periodic drive. The drive is fundamentally and topologically different. It is not clear how to analyze this latter case in relation to the phenomena of a crisis, but it is clear that this is fundamentally a new type of driving technique that nonetheless often retains some of the periodic driving behavior. Its analysis remains to be done.

V. CONCLUSIONS AND REMARKS

There are several approaches to pseudoperiodic driving, as outlined in Sec. II. We have investigated only a few with the most attention paid to addition of chaos to an existing periodic drive. Some of the theory of crises can account for the synchronization in this case, but the case of replacement of the periodic drive by a chaotic one remains a problem for analysis. We contend here as we have elsewhere [14] that new mathematical tools are probably necessary to analyze chaotically driven systems. These are topologically different drives from the periodic ones. However, as in the Rössler drive case some of the structure of the periodic drive occasionally remains. This may be a hint as to how to begin the analysis of pseudoperiodic driving using method (1).

We note that a few others have studied systems with drives modified similar to ours, but they focused on other issues. Among these are Kapitaniak [22], who studied systems with noise, especially driven with noisy drives; also Kornadt, Linz, and Lücke [23], who studied a map similar to the logistic map and added noise to the parameter which is quite similar to pseudoperiodic driving.

The phenomena uncovered here and the potential applications do provide interesting possibilities. There are several striking features of pseudoperiodic driving above threshold. One is robustness. The stable region spans at least an order of magnitude in ϵ for certain drives and is not sensitive to changes in the driving system, provided the drive remains chaotic. Another feature is that pseudoperiodic driving results in smooth response behavior which closely mimics (forever) the response behavior with a periodic drive. Finally, we often need add only a few percent of the chaotic signal to a periodic signal to eliminate multiple-period behavior.

Applications of pseudoperiodic driving can be considered in a general scenerio by noting that if one wants to drive a complex nonlinear system so that all subsystems are, in some way, stably synchronized, then the driving signal of choice would not be a periodic one, but rather a pseudoperiodic one.

A particular application of this might be in physiology. For some time researchers have noted the probable existence of chaotic signals in living organisms [24–26] and the possible association of pathology with periodic behavior. This latter suggestion is plausible since synchronization is important in many physiological functions (e.g., heart valves opening and closing in the right order). Pseudoperiodic driving offers an alternative, concrete explanation for physiological chaos: organisms use chaos to avoid multiple-period behavior. In this sense it might be best to use pseudoperiodic drives in implants which require nearly periodic stimulation of muscles or nerves. For example, pacemakers may be good candidates for pseudoperiodic drives.

Systems of coupled oscillations may also be good candidates for pseudoperiodic driving. Many times devices are coupled into arrays to increase the sensitivity or power output beyond what one device would provide. In many of these cases it is desirable that the devices run synchronously. An example of this is the Josephson-junction superconducting device arrays which have been suggested as voltage standards, oscillators, and detectors. Should the devices be nonlinear (as the Josephson-junction devices are) one runs the risk of an unsynchronized system (which is known to happen with the Josephson-junction devices). Pseudoperiodic driving may be advantageous in this situation.

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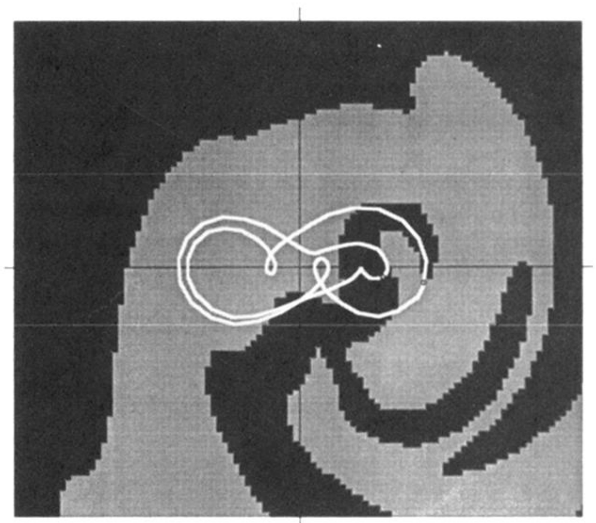

FIG. 2. Basin of attraction for the period-2 attractor PLD system (see text) along with the attractor. The black areas all synchronize with the point on the attractor in the black region. The gray areas all synchronize with the point on the attractor in the gray region.